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Weak convergence of finite-dimensional distributions of the number of empty boxes in the Bernoulli sieve

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Abstract

The Bernoulli sieve is a random allocation scheme obtained by placing independent points with the uniform $[0,1]$ law into the intervals made up by successive positions of a multiplicative random walk with factors taking values in the interval $(0,1)$. Assuming that the number of points is equal to n we investigate the weak convergence, as $n \to \infty$, of finite-dimensional distributions of the number of empty intervals within the occupancy range. A new argument enables us to relax the constraints imposed in previous papers on the distribution of the factor of the multiplicative random walk.

Key words: Bernoulli sieve, Karlin's occupancy scheme in random environment, Poissonization, weak convergence of finite-dimensional distributions

1 Introduction and main results

Let $T := \left(T_k\right)_{k\in\mathbb{N}_0}$ be a multiplicative random walk defined by

$$T_0 := 1, \quad T_k := \prod_{i=1}^{k} W_i, \quad k \in \mathbb{N} := \mathbb{N}_0 \backslash \{0\},$$

where $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, $\left(W_k\right)_{k\in\mathbb{N}}$ are independent copies of a random variable W taking values in the open interval $(0,1)$. Also, let $\left(U_k\right)_{k\in\mathbb{N}}$ be independent random variables which are independent of T and have the uniform $[0,1]$ law. A random allocation scheme in which 'balls' U_1, U_2 etc. are allocated over an infinite array of 'boxes' $(T_k, T_{k-1}]$, $k \in \mathbb{N}$, is called the *Bernoulli sieve*. The study of this allocation scheme was initiated in [6]. Since then a number of papers [8, 9, 10, 11, 12, 15, 16] has appeared which analyze some asymptotic properties of the Bernoulli sieve.

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Since a particular ball falls into the box $(T_k, T_{k-1}]$ with a random probability

$$P_k := T_{k-1} - T_k = W_1 W_2 \cdots W_{k-1}(1 - W_k),$$

the Bernoulli sieve is also the classical Karlin's allocation scheme [7, 18] with the *random frequencies* $(P_k)_{k\in\mathbb{N}}$ (or in the random environment $\big(P_k\big)$ or $\big(W_k\big)$). In this setting it is assumed that, given the environment $\big(P_k\big)$, some *abstract* balls are allocated over an infinite collection of *abstract* boxes $1, 2, \ldots$ independently with probability P_j of hitting box j. In the sequel, we say that the box $(T_k, T_{k-1}]$ has index k.

Recall that some infinite random allocation schemes in *nonrandom environment* were also investigated in [20, 22, 23]. It should be emphasized that infinite allocation schemes radically differ from the classical allocation scheme with finitely many positive frequencies (see monograph [19] for more detail).

Assuming that the number of balls to be allocated equals n (in other words, using a sample of size n from the uniform distribution on $[0, 1]$), denote by K_n the number of occupied boxes and by M_n the index of the last occupied box. Set $L_n := M_n - K_n$ and note that L_n equals the number of empty boxes with indices not exceeding M_n. The articles mentioned in the first paragraph give a fairly complete account of *one-dimensional convergence* of K_n, M_n and L_n. The present paper contains the first results concerning weak convergence of *finite-dimensional* distributions of elements of the collection $\big(L_n\big)$.

Before formulating the main results of the paper we recall an assertion given in Theorem 1.1 [16].

Proposition 1.1. *If* $\mathbb{E}|\log W| = \infty$ *and*

$$\lim_{n\to\infty} \frac{\mathbb{E}(1-W)^n}{\mathbb{E}W^n} = c \in (0, \infty),$$

then

$$L_n \ \overset{d}{\to}\ L, \quad n \to \infty, \tag{1}$$

where L is a random variable with a geometric law

$$\mathbb{P}\{L = k\} = \frac{c}{c+1}\left(\frac{1}{c+1}\right)^k, \quad k \in \mathbb{N}_0.$$

In particular, relation (1) *holds if*

$$\lim_{x\to\infty} \frac{\mathbb{P}\{|\log W| > x\}}{\mathbb{P}\{|\log(1-W)| > x\}} = c. \tag{2}$$

There are no reasons to expect that the conditions $\mathbb{E}|\log W| = \infty$ and (2) alone are sufficient for weak convergence of some *finite-dimensional* distributions related to $\big(L_n\big)$. Nevertheless, a result of this sort is given in Theorem 1.2 below under an additional assumption imposed on the *decay rate* to zero of the numerator in (2).

Let $N_\infty^{(\alpha,\,c)} := \sum_k \varepsilon_{(t_k,\, j_k)}$ be a Poisson random measure on $[0, \infty) \times (0, \infty]$ with mean measure $\mathbb{LEB} \times \nu_{\alpha,\,c}$, where $\mathbb{LEB}$ is the Lebesgue measure on $[0, \infty)$, and $\nu_{\alpha,\,c}$ is a measure on $(0, \infty]$ defined by

$$\nu_{\alpha,\,c}\big((x, \infty]\big) = c^{-1}x^{-\alpha}, \quad x > 0.$$

Further, let $\big(X_\alpha(t)\big)_{t\geq 0}$ be an α-stable subordinator which is independent of $N_\infty^{(\alpha,\,c)}$ and has the Laplace transform

$$\mathbb{E}\exp(-zX_\alpha(t)) = \exp(-\Gamma(1-\alpha)tz^\alpha), \quad z \geq 0,$$

where $\Gamma(\cdot)$ is the gamma function. Denote by $\big(X_\alpha^{\leftarrow}(s)\big)_{s\geq 0}$ an inverse α-stable subordinator defined by

$$X_\alpha^{\leftarrow}(s) := \inf\{t \geq 0 : X_\alpha(t) > s\}, \quad s \geq 0.$$

We stipulate hereafter that ℓ, $\widehat{\ell}$ and ℓ^* denote functions slowly varying at infinity. Besides, we write $Z_t(u) \overset{\text{f.d.}}{\Rightarrow} Z(u)$, $t \to \infty$ to denote weak convergence of finite-dimensional distributions meaning that for any $n \in \mathbb{N}$ and any selection $0 < u_1 < u_2 < \ldots < u_n < \infty$

$$\big(Z_t(u_1), \ldots, Z_t(u_n)\big) \overset{d}{\to} \big(Z(u_1), \ldots, Z(u_n)\big), \quad t \to \infty.$$

Theorem 1.2. *If there exist $\alpha \in (0,1)$, $c \in (0,\infty)$ and a function ℓ such that*

$$\mathbb{P}\{|\log W| > x\} \quad \sim \quad c\,\mathbb{P}\{|\log(1-W)| > x\} \ \sim \ x^{-\alpha}\ell(x), \quad x \to \infty, \tag{3}$$

then

$$L_{[e^{ut}]} \overset{\text{f.d.}}{\Rightarrow} \sum_k 1_{\{X_\alpha(t_k) \leq u < X_\alpha(t_k) + j_k\}} =: R_{\alpha,\,c}(u), \quad t \to \infty.$$

Furthermore, with $u > 0$ fixed, the distribution of $R_{\alpha,\,c}(u)$ is geometric with the success probability $c(c+1)^{-1}$.

Remark 1.3. The weak convergence of finite-dimensional distributions stated in Theorem 1.2 immediately implies the strict stationarity of the process $\big(R_{\alpha,\,c}(e^t)\big)_{t\in\mathbb{R}}$.

Theorem 1.4 and Theorem 1.5 given below refine Theorem 1.1 [15] and Theorem 1.2 [16], respectively, which deal with one-dimensional convergence only.

Theorem 1.4. *Suppose that there exist $0 \leq \beta \leq \alpha < 1$ ($\alpha + \beta > 0$) and functions ℓ and $\widehat{\ell}$ such that*

$$\mathbb{P}\{|\log W| > x\} \ \sim \ x^{-\alpha}\ell(x) \quad \textit{and} \quad \mathbb{P}\{|\log(1-W)| > x\} \ \sim \ x^{-\beta}\widehat{\ell}(x), \quad x \to \infty. \tag{4}$$

If $\alpha = \beta$, assume additionally that

$$\lim_{x\to\infty} \frac{\mathbb{P}\{|\log W| > x\}}{\mathbb{P}\{|\log(1-W)| > x\}} = 0$$

and that there exists a nondecreasing function $u(x)$ satisfying

$$\lim_{x\to\infty} \frac{\mathbb{P}\{|\log W| > x\}u(x)}{\mathbb{P}\{|\log(1-W)| > x\}} = 1.$$

Then

$$\frac{\mathbb{P}\{|\log W| > t\}}{\mathbb{P}\{|\log(1-W)| > t\}} L_{[e^{ut}]} \overset{\text{f.d.}}{\Rightarrow} \int_{[0,\,u]} (u-s)^{-\beta} \mathrm{d}X_\alpha^{\leftarrow}(s) =: W_{\alpha,\beta}(u), \quad t \to \infty. \tag{5}$$

Theorem 1.5. *Suppose that there exist $\beta \in [0,1)$ and a function $\widehat{\ell}$ such that*

$$\mathbb{P}\{|\log(1-W)| > x\} \;\sim\; x^{-\beta}\widehat{\ell}(x), \quad x \to \infty. \tag{6}$$

(a) *If $\sigma^2 = \mathrm{Var}\,(\log W) < \infty$ then*

$$\frac{L_{[e^{ut}]} - \mu^{-1}\int_0^{ut}\mathbb{P}\{|\log(1-W)| > y\}\mathrm{d}y}{\sqrt{\mu^{-1}\int_0^t \mathbb{P}\{|\log(1-W)| > y\}\mathrm{d}y}} \;\overset{\text{f.d.}}{\Rightarrow}\; V(u), \quad t \to \infty, \tag{7}$$

where $\mu := \mathbb{E}|\log W| < \infty$, and $\big(V(s)\big)_{s\geq 0}$ is a centered Gaussian process with

$$\mathbb{E}V(t)V(s) = t^{1-\beta} - (t-s)^{1-\beta}, \quad 0 \leq s \leq t.$$

(b) *Suppose that $\sigma^2 = \infty$ and there exists a function ℓ such that*

$$\int_{[0,\,x]} y^2 \mathbb{P}\{|\log W| \in \mathrm{d}y\} \;\sim\; \ell(x), \quad x \to \infty. \tag{8}$$

Let $c(x) = x^{1/2}\ell^*(x)$ be a positive function satisfying $\lim_{x\to\infty} x\ell(c(x))/c^2(x) \;=\; 1$.
(b1) *If*

$$\lim_{x\to\infty}\mathbb{P}\{|\log(1-W)| > x\}\big(\ell^*(x)\big)^2 = 0 \tag{9}$$

then relation (7) holds true.
(b2) *If $\beta = 0$ and, in addition,*

$$\lim_{x\to\infty}\mathbb{P}\{|\log(1-W)| > x\}\big(\ell^*(x)\big)^2 = \infty$$

then

$$\frac{L_{[e^{ut}]} - \mu^{-1}\int_0^{ut}\mathbb{P}\{|\log(1-W)| > y\}\mathrm{d}y}{\mu^{-3/2}c(t)\mathbb{P}\{|\log(1-W)| > t\}} \;\overset{\text{f.d.}}{\Rightarrow}\; \int_{[0,\,u]}(u-s)^{-\beta}\mathrm{d}Z_2(s) =: W_{2,\,\beta}(u), \quad t \to \infty,$$

where $\big(Z_2(s)\big)_{s\geq 0}$ is a Brownian motion.
(c) *Suppose that there exist $\alpha \in (1,2)$ and a function ℓ such that*

$$\mathbb{P}\{|\log W| > x\} \;\sim\; x^{-\alpha}\ell(x), \quad x \to \infty. \tag{10}$$

Let $c(x) = x^{1/\alpha}\ell^*(x)$ be a positive function satisfying $\lim_{x\to\infty} x\ell(c(x))/c^\alpha(x) \;=\; 1$.
(c1) *If*

$$\lim_{x\to\infty}\mathbb{P}\{|\log(1-W)| > x\}x^{2/\alpha-1}\big(\ell^*(x)\big)^2 = 0 \tag{11}$$

then relation (7) holds true.
(c2) *Suppose that $\beta \in [0, 2/\alpha - 1]$. If $\beta = 2/\alpha - 1$, assume additionally that*

$$\lim_{x\to\infty}\mathbb{P}\{|\log(1-W) > x|\}x^{2/\alpha-1}\big(\ell^*(x)\big)^2 = \infty. \tag{12}$$

Then

$$\frac{L_{[e^{ut}]}-\mu^{-1}\int_0^{ut}\mathbb{P}\{|\log(1-W)|>y\}\mathrm{d}y}{\mu^{-1-1/\alpha}c(t)\mathbb{P}\{|\log(1-W)|>t\}} \overset{\text{f.d.}}{\Rightarrow} \int_{[0,\,u]}(u-s)^{-\beta}\mathrm{d}Z_\alpha(s) =: W_{\alpha,\,\beta}(u), \quad t\to\infty,$$

where $\big(Z_\alpha(s)\big)_{s\geq 0}$ *is an* α*-stable Lévy process with*

$$\mathbb{E}\exp\{izZ_\alpha(1)\} = \exp\{-|z|^\alpha\Gamma(1-\alpha)(\cos(\pi\alpha/2)+\mathrm{i}\sin(\pi\alpha/2)\operatorname{sign}(z))\},\ z\in\mathbb{R}. \quad (13)$$

Remark 1.6. (I) Existence and the properties of functions $c(t)$ claimed in parts (b) and (c) of Theorem 1.5 are well-known. For instance, a function $c(t)$ in part (b) is an asymptotic inverse to $t^2/\int_{[0,\,t]}y^2\mathrm{d}\mathbb{P}\{|\log W|\in\mathrm{d}y\}\sim t^2/\ell(t)$. Consequently, according to Proposition 1.5.15 [3], $c(t)\sim t^{1/2}(L^{\#}(t))^{1/2}$, where $L^{\#}(t)$ is the de Bruijn conjugate of $1/\ell(t^{1/2})$.
(II) Suppose that the distribution of $|\log W|$ is nonlattice. As shown in Theorem 1.2 [16], the weak convergence of one-dimensional distributions in parts (a), (b1) and (c1) of Theorem 1.5 does not require the regular variation of $\mathbb{P}\{|\log(1-W)|>x\}$. It suffices to assume that $\mathbb{E}|\log(1-W)|=\infty$ along with all the other assumptions of the theorem.
(III) Let $\big(Z_2(s)\big)_{s\geq 0}$ be a Brownian motion, independent of $\big(V(s)\big)$. A. Yu. Pilipenko attracted our attention to the fact that finite-dimensional distributions of the process $\big(V(s)+Z_2(s^{1-\beta})\big)$ (recall that $\beta\in[0,1)$) coincide with those of a scaled *fractional Brownian motion*. Clearly, if $\beta=0$ then finite-dimensional distributions of $\big(V(s)\big)$ coincide with those of a Brownian motion.

Two situations, though worth investigating, are ruled out in the present paper.
(I) Suppose that the assumptions of part (b) or (c) of Theorem 1.5 hold, and the limits in relation (9) or (11), respectively, are finite and nonzero. The authors do not know whether there is even the one-dimensional convergence in these cases.
(II) The theorems just formulated are collected together as their proofs follow the same approach. Unfortunately, such an approach is not applicable to multiplicative random walks with $\mathbb{E}(|\log W|+|\log(1-W)|)<\infty$. For this reason we do not treat this case here. Under the latter assumption two different proofs of *one-dimensional convergence* of the number of empty boxes can be found in [11, 12].

The structure of the paper is as follows. In the context of problems related to random allocations a Poissonization which is a transition from the original scheme with deterministic number of balls to a scheme with random (Poisson) number of balls is a rather efficient tool. In Section 2 we formulate three lemmas which are proved in Sections 3 and 4, respectively. While two of these indicate that the Poissonization of the Bernoulli sieve is expedient, the third takes care of a de-Poissonization, i.e., a reverse transition. The proofs of Theorems 1.2, 1.4 and 1.5 are given in Sections 5, 6 and 7, respectively. Finally, some auxiliary results are collected in the Appendix.

2 Poissonization and de-Poissonization

Let $\big(\tau_k\big)_{k\in\mathbb{N}}$ be a Poisson flow with unit intensity which is independent of the random variables $\big(U_j\big)$ and the multiplicative random walk T. Denote by $\big(\pi(t)\big)_{t\geq 0}$ the corresponding

Poisson process defined by

$$\pi(t) := \#\{k \in \mathbb{N} : \tau_k \le t\}, \quad t \ge 0.$$

Instead of the scheme with n balls we will work with a *Poissonized* version of the Bernoulli sieve in which, for $j \in \mathbb{N}$, the jth ball (the point U_j) is thrown in the boxes (the intervals $(T_{k-1}, T_k]$) at the epoch τ_j. Thus, the random number $\pi(t)$ of balls will be allocated over the boxes within $[0, t]$. Denote by $\pi_k(t)$ the number of balls which fall into the kth box within $[0, t]$. It is evident that, given the collection (environment) $\big(T_j\big)$, (1) the process $\big(\pi_k(t)\big)_{t \ge 0}$ is, for each k a Poisson process with intensity $P_k = T_{k-1} - T_k$, and (2) these processes are *independent* for different k. It is this latter property which demonstrates the advantage of the Poissonized scheme over the original one.

Put $M(t) := M_{\pi(t)}$, $K(t) := K_{\pi(t)}$ and $L(t) := L_{\pi(t)}$. With this notation in view $L(t)$ is the number of empty boxes within the occupancy range obtained by throwing $\pi(t)$ balls. Recall that the Bernoulli sieve can be interpreted as the Karlin's allocation scheme in the random environment $\big(W_k\big)$ which is given by i.i.d. random variables. The first two auxiliary results of the present paper reveal that one can investigate the asymptotics of a relatively simple functional which is determined by the environment only rather than that of $L(t)$.

Denote by $\big(S_n\big)_{n \in \mathbb{N}_0}$ a zero-delayed random walk defined by

$$S_0 := 0, \quad S_n := |\log W_1| + \ldots + |\log W_n|, \quad n \in \mathbb{N},$$

and put $\eta_n := |\log(1 - W_n)|$.

Lemma 2.1. *If* $\mathbb{E}|\log W| = \infty$, *then*

$$L(e^{ut}) - \sum_{k \ge 0} 1_{\{S_k \le ut < S_k + \eta_{k+1}\}} \overset{\text{f.d.}}{\Rightarrow} 0, \quad t \to \infty. \tag{14}$$

Lemma 2.2. *If* $\mathbb{E}|\log W| < \infty$, *then*

$$\frac{L(e^{ut}) - \sum_{k \ge 0} 1_{\{S_k \le ut < S_k + \eta_{k+1}\}}}{a(t)} \overset{\text{f.d.}}{\Rightarrow} 0, \quad t \to \infty \tag{15}$$

for any function $a(t)$ *satisfying* $\lim_{t \to \infty} a(t) = \infty$. *In other words, finite-dimensional distributions of the process* $\big(L(e^{ut}) - \sum_{k \ge 0} 1_{\{S_k \le ut < S_k + \eta_{k+1}\}},\, t \ge 0\big)$ *are tight.*

Lemma 2.3 given below allows us to implement a *de-Poissonization*, i.e., a reverse transition from the scheme with Poisson number of balls to the original scheme with deterministic number of balls. Although the papers [11, 15, 16] offer several approaches to the de-Poissonization of the number of empty boxes, the result of Lemma 2.3 is the strongest one out of those known to the authors, even if only one-dimensional distributions are considered.

Lemma 2.3. *With no assumptions on the expectation of* $|\log W|$,

$$L(e^{ut}) - L_{[e^{ut}]} \overset{\text{f.d.}}{\Rightarrow} 0, \quad t \to \infty.$$

3 Proof of Lemma 2.1 and Lemma 2.2

Put

$$\nu(t) := \inf\{k \in \mathbb{N} : S_k > t\}, \quad t \geq 0,$$

and denote by U the renewal function generated by the random walk $S_k, k \geq 0$, i.e.,

$$U(t) := \mathbb{E}\nu(t) = \sum_{k \geq 0} \mathbb{P}\{S_k \leq t\}, \quad t \geq 0.$$

In the sequel we repeatedly use the Blackwell theorem and the key renewal theorem. Since $\mathbb{E}|\log W| = \infty$, a separate treatment of the situation when the distribution of $|\log W|$ is lattice is not needed. Indeed, using the monotonicity of U and appealing to the Blackwell theorem we conclude that in the lattice case as well as in the case when the distribution of $|\log W|$ is nonlattice,

$$\lim_{t \to \infty} \big(U(t+h) - U(t)\big) = 0, \tag{16}$$

for any $h > 0$. Thus repeating almost literally the proof of the key renewal theorem given in [25], p. 241 we conclude that

$$\lim_{t \to \infty} \int_{[0,\, t]} g(t-x) \mathrm{d}U(x) = 0,$$

provided that g is a function directly Riemann integrable on $[0, \infty)$, and

$$\lim_{t \to \infty} \int_{[t,\, \infty)} g(t-x) \mathrm{d}U(x) = 0,$$

provided that g is directly Riemann integrable on $(-\infty, 0]$.

Proof of Lemma 2.1. It suffices to prove that the left-hand side of relation (14) with $u = 1$ converges to zero in probability and to use the Cramér-Wold device. To simplify understanding we divide the proof into several steps.

Step 1. We intend to show that the maximal index of boxes discovered by the Poisson process within $[0, e^t]$ satisfies

$$M(e^t) - \nu(t) \overset{P}{\to} 0, \quad t \to \infty.$$

To this end, put $E(n) := -\log\min(U_1, \ldots, U_n)$ and note that $M(e^t) = \nu(E(\pi(e^t)))$. As $n \to \infty$, the difference $E(n) - \log n$ converges in distribution to a random variable E^* obeying the Gumbel distribution. Since the sequence $\big(E(n)\big)$ is independent of the process $\big(\pi(t)\big)$, the difference $E(\pi(e^t)) - \log \pi(e^t)$ converges in distribution, as $t \to \infty$, to E^*, as well. By the weak law of large numbers for Poisson processes, $\log \pi(e^t) - t \overset{P}{\to} 0$, as $t \to \infty$. Hence

$$\lim_{t \to \infty} \big(E(\pi(e^t)) - t\big) = E^* \ \text{ in distribution.} \tag{17}$$

For brevity, set $R(t) := E(\pi(e^t))$. Using Markov's inequality and the fact that the renewal function $U(t)$ is nondecreasing we obtain

$$\begin{aligned}
\mathbb{P}\Big\{\big(\nu(R(t)) - \nu(t)\big)1_{\{0<R(t)-t\leq\gamma\}} > \varepsilon \Big| R(t)\Big\} &\leq \varepsilon^{-1}\mathbb{E}\Big(\big(\nu(R(t)) - \nu(t)\big)1_{\{0<R(t)-t\leq\gamma\}} \Big| R(t)\Big) \\
&= \varepsilon^{-1}\big(U\big(t + R(t) - t\big) - U(t)\big)1_{\{0<R(t)-t\leq\gamma\}} \\
&\leq \varepsilon^{-1}\big(U(t+\gamma) - U(t)\big),
\end{aligned}$$

for any $\gamma > 0$ and $\varepsilon > 0$. This combined with (16) yields

$$\lim_{t\to\infty} \mathbb{P}\Big\{ \big(\nu(R(t)) - \nu(t)\big) 1_{\{0<R(t)-t\le\gamma\}} > \varepsilon \Big| R(t) \Big\} = 0 \ \text{ almost surely.}$$

Consequently

$$\big(\nu(R(t)) - \nu(t)\big) 1_{\{0<R(t)-t\le\gamma\}} \stackrel{P}{\to} 0, \quad t\to\infty,$$

by the Lebesgue dominated convergence theorem.

Recalling (17) and using the absolute continuity of the law of E^* and the inequality

$$\mathbb{P}\big\{ \big(\nu(R(t)) - \nu(t)\big) 1_{\{R(t)-t>\gamma\}} > \varepsilon \big\} \le \mathbb{P}\{R(t) - t > \gamma\},$$

which holds for any $\gamma > 0$ and $\varepsilon > 0$, we see that

$$\overline{\lim_{t\to\infty}}\, \mathbb{P}\big\{ \big(\nu(R(t)) - \nu(t)\big) 1_{\{R(t)-t>\gamma\}} > \varepsilon \big\} \le \mathbb{P}\{E^* > \gamma\}$$

and therefore,

$$\lim_{\gamma\to\infty} \overline{\lim_{t\to\infty}}\, \mathbb{P}\big\{ \big(\nu(R(t)) - \nu(t)\big) 1_{\{R(t)-t>\gamma\}} > \varepsilon \big\} = 0.$$

The estimates above lead to an important relation

$$\big(\nu(R(t)) - \nu(t)\big) 1_{\{R(t)-t>0\}} \stackrel{P}{\to} 0, \quad t\to\infty.$$

Arguing similarly we arrive at

$$\big(\nu(R(t)) - \nu(t)\big) 1_{\{R(t)-t\le 0\}} \stackrel{P}{\to} 0, \quad t\to\infty.$$

Step 2. We are seeking a good approximation for $K(e^t)$ the number of boxes discovered by the Poisson process within $[0, e^t]$. More precisely, we prove that

$$K(e^t) - \sum_{k\ge 0} \Big(1 - \exp\big(- e^{t-S_k}(1 - W_{k+1})\big)\Big) \stackrel{P}{\to} 0, \quad t\to\infty.$$

We start with the representation

$$K(e^t) = \sum_{k\ge 1} 1_{\{\pi_k(e^t)\ge 1\}}, \tag{18}$$

where $\pi_k(e^t)$ is the number of balls (in the Poissonized scheme) landing in the kth box within $[0,\, e^t]$. In view of

$$\mathbb{E}\big(K(e^t)|(P_j)\big) = \sum_{k\ge 0} \Big(1 - \exp\big(- e^{t-S_k}(1 - W_{k+1})\big)\Big), \tag{19}$$

to establish the desired approximation it is sufficient to prove that

$$\lim_{t\to\infty} \mathbb{E}\,\mathrm{Var}\big(K(e^t)|(P_j)\big) = 0. \tag{20}$$

Given $\big(P_j\big)$, the indicators in (18) are independent. Hence

$$\begin{aligned}\mathbb{E}\,\mathrm{Var}\big(K(e^t)|(P_j)\big) &= \mathbb{E}\sum_{k\geq 0}\Big(\exp\big(-e^{t-S_k}(1-W_{k+1})\big)-\exp\big(-2e^{t-S_k}(1-W_{k+1})\big)\Big)\\ &= \int_{[0,\infty)}\Big(\varphi(e^{t-y})-\varphi(2e^{t-y})\Big)\mathrm{d}U(y),\end{aligned}$$

where $\varphi(y) := \mathbb{E}e^{-y(1-W)}$. By Lemma 8.1 in the Appendix, $g_0(y) := \varphi(e^y)-\varphi(2e^y)$ is a directly Riemann integrable function on $\mathbb{R}$. Applying now the key renewal theorem justifies relation (20).

STEP 3. We intend to prove the relation

$$Z(t) := \sum_{k\geq 0}\Big(1-\exp\big(-e^{t-S_k}(1-W_{k+1})\big)\Big)1_{\{S_k>t\}} \overset{P}{\to} 0,\quad t\to\infty.$$

According to Lemma 8.1, $g_1(y) := \mathbb{E}\big(1-\exp\big(-e^y(1-W)\big)\big)$ is a directly Riemann integrable function on $(-\infty, 0]$. Hence

$$\mathbb{E}Z(t) = \int_{[t,\infty)} g_1(t-y)\mathrm{d}U(y) \ \to\ 0,\quad t\to\infty,$$

by the key renewal theorem.

STEP 4. We are going to prove the relation

$$Y(t) := \sum_{k\geq 0}\Big(\exp\big(-e^{t-S_k}(1-W_{k+1})\big)-1_{\{S_k+\eta_{k+1}>t\}}\Big)1_{\{S_k\leq t\}} \overset{P}{\to} 0,\quad t\to\infty.$$

To this end, write $Y(t)$ as the difference of two nonnegative random functions

$$\begin{aligned}Y(t) &= \sum_{k\geq 0}\exp\big(-e^{t-S_k}(1-W_{k+1})\big)1_{\{S_k+\eta_{k+1}\leq t\}}\\ &- \sum_{k\geq 0}\Big(1-\exp\big(-e^{t-S_k}(1-W_{k+1})\big)\Big)1_{\{S_k\leq t<S_k+\eta_{k+1}\}} =: Y_1(t)-Y_2(t)\end{aligned}$$

and show that $\lim_{t\to\infty}\mathbb{E}Y_i(t) = 0$, $i=1,2$. Indeed, according to Lemma 8.1, the functions $g_2(y) := \mathbb{E}\exp\big(-e^y(1-W)\big)1_{\{1-W>e^{-y}\}}$ and $g_3(y) := \mathbb{E}(1-\exp(-e^y(1-W)))1_{\{1-W\leq e^{-y}\}}$ are directly Riemann integrable on $[0,\infty)$. Hence, by the key renewal theorem,

$$\mathbb{E}Y_1(t) = \int_{[0,\,t]} g_2(t-y)\mathrm{d}U(y) \ \to\ 0,\quad t\to\infty$$

and

$$\mathbb{E}Y_2(t) = \int_{[0,\,t]} g_3(t-y)\mathrm{d}U(y) \ \to\ 0,\quad t\to\infty,$$

which is the desired result.

Combining conclusions of the four steps finishes the proof of Lemma 2.1.

PROOF OF LEMMA 2.2. If the distribution of $|\log W|$ is nonlattice the proof of Lemma 2.2 proceeds along the same lines as that of Lemma 2.1. If the distribution of $|\log W|$ is l-lattice, for some $l>0$, an additional argument is only needed for the step 1 of the proof of Lemma 2.1. To implement the steps 2 through 4 one may use Lemma 8.2 from the Appendix.

STEP 1. Fix any $\gamma>0$ and select an $m\in\mathbb{N}$ such that $\gamma\le ml$. With this γ and $\varepsilon>0$ in hands, we use the inequality

$$\mathbb{P}\bigg\{\frac{\nu(R(t))-\nu(t)}{a(t)}1_{\{0<R(t)-t\le\gamma\}}>\varepsilon\bigg|R(t)\bigg\}\le\frac{U(t+ml)-U(t)}{\varepsilon a(t)}$$

in combination with the relation

$$\lim_{t\to\infty}\big(U(t+ml)-U(t)\big)=\frac{ml}{\mathbb{E}|\log W|}$$

and the Lebesgue bounded convergence theorem to conclude that

$$\frac{\nu(R(t))-\nu(t)}{a(t)}1_{\{0<R(t)-t\le\gamma\}}\ \overset{P}{\to}\ 0,\quad t\to\infty.$$

This completes the proof for the step 1.

4 Proof of Lemma 2.3

It suffices to check that

$$K(t)-K_{[t]}\ \overset{P}{\to}\ 0\ \text{ and }\ M(t)-M_{[t]}\ \overset{P}{\to}\ 0,\quad t\to\infty,\tag{21}$$

and to use the Cramér-Wold device. In view of the inequality $\mathbb{P}(M(t)\neq M_{[t]})\le\mathbb{P}(K(t)\neq K_{[t]})$, only the first relation in (21) needs a proof.

We first show that, for any $x>0$,

$$K(t+x\sqrt{t})-K(t-x\sqrt{t})\ \overset{P}{\to}\ 0,\quad t\to\infty.\tag{22}$$

By (19), we have, for large enough t,

$$\mathbb{E}\big(K(t+x\sqrt{t})-K(t-x\sqrt{t})\big)=\int_{[0,\infty)}\big(\varphi((t-x\sqrt{t})e^{-y})-\varphi((t+x\sqrt{t})e^{-y})\big)\mathrm{d}U(y),$$

where $\varphi(y)=\mathbb{E}e^{-y(1-W)}$. As the function $-\varphi'(y)$ is nonincreasing, we infer

$$\varphi\big((t-x\sqrt{t})e^{-y}\big)-\varphi\big((t+x\sqrt{t})e^{-y}\big)\le-\varphi'\big((t-x\sqrt{t})e^{-y}\big)2x\sqrt{t}e^{-y},$$

by the mean value theorem for differentiable functions, and therefore

$$\mathbb{E}\big(K(t+x\sqrt{t})-K(t-x\sqrt{t})\big)\le\frac{2x\sqrt{t}}{t-x\sqrt{t}}\int_{[0,\infty)}\big(-\varphi'\big((t-x\sqrt{t})e^{-y}\big)\big)(t-x\sqrt{t})e^{-y}\mathrm{d}U(y).$$

By Lemma 8.1, $g_4(y) = -\varphi'(e^y)e^y$ is a directly Riemann integrable function on $\mathbb{R}$. This and Lemma 8.2 yield

$$\int_{[0,\infty)} \big(-\varphi'\big((t-x\sqrt{t})e^{-y}\big)\big)(t-x\sqrt{t})e^{-y}\mathrm{d}U(y) = O(1), \quad t\to\infty.$$

Hence, for any $x>0$,

$$\lim_{t\to\infty}\mathbb{E}\big(K(t+x\sqrt{t})-K(t-x\sqrt{t})\big)=0,$$

which entails (22).

The process $\big(K(s)\big)_{s\geq 0}$ is almost surely nondecreasing. This implies that, for any $x>0$,

$$\begin{aligned}|K_{[t]}-K(t)| &= |K(\tau_{[t]})-K(t)|1_{\{t-x\sqrt{t}\leq\tau_{[t]}\leq t+x\sqrt{t}\}}\\ &+ |K(\tau_{[t]})-K(t)|1_{\{|\tau_{[t]}-t|>x\sqrt{t}\}}\\ &\leq K(t+x\sqrt{t})-K(t-x\sqrt{t})+|K(\tau_{[t]})-K(t)|1_{\{|\tau_{[t]}-t|>x\sqrt{t}\}}.\end{aligned}$$

Hence, for any $\varepsilon>0$,

$$\begin{aligned}\mathbb{P}\{|K_{[t]}-K(t)|>2\varepsilon\} &\leq \mathbb{P}\{K(t+x\sqrt{t})-K(t-x\sqrt{t})>\varepsilon\}\\ &+ \mathbb{P}\{|K(\tau_{[t]})-K(t)|1_{\{|\tau_{[t]}-t|>x\sqrt{t}\}}>\varepsilon\}\\ &\leq \mathbb{P}\{K(t+x\sqrt{t})-K(t-x\sqrt{t})>\varepsilon\}+\mathbb{P}\{|\tau_{[t]}-t|>x\sqrt{t}\}.\end{aligned}$$

Recalling (22) and using the central limit theorem give

$$\overline{\lim_{t\to\infty}}\mathbb{P}\{|K_{[t]}-K(t)|>2\varepsilon\}\leq\mathbb{P}\{|\mathcal{N}(0,1)|>x\},$$

where $\mathcal{N}(0,1)$ stands for a random variable with the standard normal law. Letting $x\to\infty$ establishes the first relation in (21). The proof is complete.

5 Proof of Theorem 1.2

According to Lemma 2.3, it suffices to show that

$$L(e^{ut}) \overset{\text{f.d.}}{\Rightarrow} R_{\alpha,c}(u), \quad t\to\infty.$$

Condition (3) entails $\mathbb{E}|\log W|=\infty$. Hence applying Lemma 2.1 we conclude that the desired assertion is equivalent to

$$\sum_{k\geq 0}1_{\{S_k\leq ut<S_k+\eta_{k+1}\}} \overset{\text{f.d.}}{\Rightarrow} R_{\alpha,c}(u), \quad t\to\infty. \tag{23}$$

We introduce more notation to be used in this section and Lemma 8.4 in the Appendix: $D:=D[0,\infty)$ – the Skorohod space of right-continuous real-valued functions on $[0,\infty)$ which have finite limits from the left on $(0,\infty)$;

$M_p([0,\infty)\times(0,\infty])$ – the set of Radon point measures on $[0,\infty)\times(0,\infty]$ endowed with the vague topology;
$C_K([0,\infty)\times(0,\infty])$ – the set of nonnegative continuous functions on $[0,\infty)\times(0,\infty]$ with compact support[1];
$\mu_{\alpha,c}$ – a measure on $(0,\infty]\times(0,\infty]$ such that

$$\mu_{\alpha,c}\big\{(u,v):u>x_1\ \text{ or }\ v>x_2\big\}=x_1^{-\alpha}+c^{-1}x_2^{-\alpha},\quad x_1x_2>0,$$

here the constant c is the same as in (3).

Also, recall the notation $N_\infty^{(\alpha,c)}$, $\nu_{\alpha,c}$, $\big(X_\alpha(t)\big)$ and $\big(S_n\big)$ introduced in the paragraphs preceding Theorem 1.2 and Lemma 2.1, respectively.

It is well-known that the condition $\mathbb{P}\{|\log W|>x\}\sim x^{-\alpha}\ell(x)$ with $\alpha\in(0,1)$ ensures

$$\frac{S_{[ut]}}{c(t)}\ \Rightarrow\ X_\alpha(u),\quad t\to\infty$$

on D equipped with the Skorohod J_1-topology, where $c(t)$ is any positive function satisfying $\lim_{t\to\infty} tc^{-\alpha}(t)\ell(c(t))=1$ (such functions do exist, see Remark 1.6(I)). Also, we have

$$\frac{S_{[ut]-1}}{c(t)}\ \Rightarrow\ X_\alpha(u),\quad t\to\infty \tag{24}$$

under the J_1-topology on D, where $S_{[ut]-1}=0$ for $0\le u<1/t$. Indeed, according to Theorem 3 [2], the obvious weak convergence of finite-dimensional distributions entails the weak convergence under the J_1-topology, as, for each $t>0$, paths of the process in the left-hand side of (24) are nondecreasing almost surely, and the limiting subordinator is stochastically continuous. Further, according to Proposition 3.21 [24], the condition $\mathbb{P}\{|\log(1-W)|>x\}\sim c^{-1}x^{-\alpha}\ell(x)$ entails the convergence

$$\sum_{k\ge1}\varepsilon_{(k/t,\,\eta_k/c(t))}\ \Rightarrow\ N_\infty^{(\alpha,c)},\quad t\to\infty$$

in $M_p([0,\infty)\times(0,\infty])$ equipped with the vague topology. By the definition of vague convergence the latter is equivalent to the following one-dimensional convergence

$$\sum_{k\ge1}g\big(k/t,\,\eta_k/c(t)\big)\ \overset{d}{\to}\ \sum_m g(t_m,\,j_m),\quad t\to\infty \tag{25}$$

for any $g\in C_K([0,\infty)\times(0,\infty])$.

If the jumps of random walk had the same law as $|\log W|$ and were independent of the sequence $\big(\eta_k\big)=\big(|\log(1-W_k)|\big)$ then relation (23) would follow from Corollary 2.3 [21]. In the present situation where the aforementioned independence is absent the proof given by Mikosch and Resnick still applies except that the relation

$$\left(\frac{S_{[ut]-1}}{c(t)},\ \sum_{k\ge1}\varepsilon_{(k/t,\,\eta_k/c(t))}\right)\ \Rightarrow\ (X_\alpha(u),N_\infty^{(\alpha,c)}),\quad t\to\infty \tag{26}$$

[1]We alert the reader that the roles of 0 and ∞ must be interchanged for the second coordinate so that the sets $[0,a]\times[b,\infty]$ are compact for $a,b>0$.

on $D \times M_p([0,\infty) \times (0,\infty])$ endowed with the product topology which is generated by J_1 and vague topologies *has to be proved*, whereas in the situation treated in [21] (26) automatically follows. Indeed, if relation (26) holds true then repeating literally the last fragment of the proof of Theorem 2.1 [21] enables us to infer

$$\sum_{k\geq 0} \varepsilon_{(S_k/c(t),\, \eta_{k+1}/c(t))} \ \Rightarrow \ \sum_m \varepsilon_{(X_\alpha(t_m),\, j_m)}, \quad t\to\infty$$

on $M_p([0,\infty)\times(0,\infty])$. A transition from the last relation to (23) may be implemented along the lines of the proof of Corollary 2.3 [21].

According to Lemma 8.4, relation (26) follows if we can prove

$$\Big(\frac{S_{[ut]-1}}{c(t)}, \sum_{k\geq 1} g\big(k/t,\, \eta_k/c(t)\big)\Big) \ \Rightarrow \ \big(X_\alpha(u), \sum_m g(t_m,\, j_m)\big), \quad t\to\infty$$

on $D\times[0,\infty)$, for any function $g\in C_K([0,\infty)\times(0,\infty])$.

Considering the second coordinates as functions in D which take constant values and appealing to Lemma 2.6 [14] we conclude that it suffices to prove the weak convergence of finite-dimensional distribution, i.e.,

$$\sum_{i=1}^n \gamma_i \frac{S_{[u_it]-1}}{c(t)} + \gamma \sum_{k\geq 1} g\big(k/t, \eta_k/c(t)\big) \ \stackrel{d}{\to} \ \sum_{i=1}^n \gamma_i X_\alpha(u_i) + \gamma\sum_m g(t_m,\, j_m), \quad t\to\infty \tag{27}$$

for any $n\in\mathbb{N}$, any $\gamma,\gamma_1,\ldots,\gamma_n\geq 0$ and any $0<u_1<u_2<\ldots<u_n$, as well as the tightness of the coordinates.

The tightness of the coordinates follows from (24) and (25), respectively. By technical reasons it is more convenient to check the relation

$$\sum_{i=1}^n \gamma_i \frac{S_{[u_it]}}{c(t)} + \gamma \sum_{k\geq 1} g\big(k/t, \eta_k/c(t)\big) \ \stackrel{d}{\to} \ \sum_{i=1}^n \gamma_i X_\alpha(u_i) + \gamma\sum_m g(t_m,\, j_m), \quad t\to\infty \tag{28}$$

which is equivalent to (27), by Slutsky's lemma.

Let us check (28) by applying the method used to prove Proposition 3.21 [24]. For $z>0$, we have

$$\begin{aligned}
\phi_t(z) \ &:= \ \mathbb{E}\exp\Big(-z\Big(\sum_{i=1}^n \gamma_i\frac{S_{[u_it]}}{c(t)} + \gamma\sum_{k\geq 1} g\big(k/t,\, \eta_k/c(t)\big)\Big)\Big)\\
&= \ \mathbb{E}\exp\Big(-z\Big(\sum_{i=1}^n \frac{\gamma_i}{c(t)}\sum_{k\geq 1} |\log W_k| 1_{\{k\leq u_it\}} + \gamma\sum_{k\geq 1} g\big(k/t, \eta_k/c(t)\big)\Big)\Big)\\
&= \ \prod_{k\geq 1}\mathbb{E}\exp\Big(-z\Big(\frac{|\log W|}{c(t)}\sum_{i=1}^n \gamma_i 1_{\{k\leq u_it\}} + \gamma g\big(k/t,\, |\log(1-W)|/c(t)\big)\Big)\Big)\\
&= \ \prod_{k\geq 1}\int_{[0,\infty)\times[0,\infty)} \big(1-K(z,u,v,k/t)\big)\mathbb{P}\Big\{\frac{|\log W|}{c(t)}\in \mathrm{d}u, \frac{|\log(1-W)|}{c(t)}\in\mathrm{d}v\Big\},
\end{aligned}$$

where $K(z,u,v,w) := 1-\exp\Big(-z\Big(u\sum_{i=1}^n \gamma_i 1_{\{w\le u_i\}}+\gamma g(w,v)\Big)\Big)$. Denote by C the (compact) support of g. It is clear that

$$\int_{[0,\infty)\times[0,\infty)} K(z,u,v,k/t)\mathbb{P}\Big\{\frac{|\log W|}{c(t)}\in {\rm d}u,\ \frac{|\log(1-W)|}{c(t)}\in {\rm d}v\Big\}$$
$$\le\ \mathbb{E}\Big(1-\exp\Big(-\frac{z|\log W|}{c(t)}\sum_{i=1}^n \gamma_i 1_{\{k\le u_i t\}}\Big)\Big)+\mathbb{P}\Big\{\Big(\frac{|\log W|}{c(t)},\ \frac{|\log(1-W)|}{c(t)}\Big)\in C\Big\},$$

for any $k\in\mathbb{N}$, which implies

$$\lim_{t\to\infty}\sup_{k\in\mathbb{N}}\int_{[0,\infty)\times[0,\infty)} K(z,u,v,k/t)\mathbb{P}\Big\{\frac{|\log W|}{c(t)}\in {\rm d}u,\frac{|\log(1-W)|}{c(t)}\in {\rm d}v\Big\}=0. \qquad (29)$$

Obviously, for $x_0>0$ small enough there exists $M=M(x_0)>0$ such that

$$0\le -\log(1-x)-x\le Mx^2,\qquad 0<x\le x_0.$$

This in combination with (29) gives

$$0\le -\log\phi_t(z)-\sum_{k\ge 1}\int_{[0,\infty)\times[0,\infty)} K(z,u,v,k/t)\mathbb{P}\Big\{\frac{|\log W|}{c(t)}\in {\rm d}u,\ \frac{|\log(1-W)|}{c(t)}\in {\rm d}v\Big\}$$
$$\le\ M\sum_{k\ge 1}\Big(\int_{[0,\infty)\times[0,\infty)} K(z,u,v,k/t)\mathbb{P}\Big\{\frac{|\log W|}{c(t)}\in {\rm d}u,\ \frac{|\log(1-W)|}{c(t)}\in {\rm d}v\Big\}\Big)^2,$$

for t large enough, and thereupon

$$\lim_{t\to\infty}\Big(-\log\phi_t(z)-\sum_{k\ge 1}\int_{[0,\infty)\times[0,\infty)} K(z,u,v,k/t)\mathbb{P}\Big\{\frac{|\log W|}{c(t)}\in {\rm d}u,\ \frac{|\log(1-W)|}{c(t)}\in {\rm d}v\Big\}\Big)=0, \qquad (30)$$

for each $z>0$, by another appeal to (29).

Conditions (3) entail

$$t\mathbb{P}\Big\{\Big(\frac{|\log W|}{c(t)},\ \frac{|\log(1-W)|}{c(t)}\Big)\in\cdot\Big\}\ \overset{v}{\to}\ \mu_{\alpha,c},\quad t\to\infty \qquad (31)$$

and

$$t\mathbb{P}\Big\{\frac{|\log(1-W)|}{c(t)}\in\cdot\Big\}\ \overset{v}{\to}\ \nu_{\alpha,c},\quad t\to\infty, \qquad (32)$$

where $\overset{v}{\to}$ denotes vague convergence of measures. Observe that $\mu_{\alpha,c}$ is a measure concentrated on the axes. This justifies the independence of the components of the limiting vector in (26) as well as the integration formula

$$\int_{[0,\infty)\times[0,\infty)} f(u,v)\mu_{\alpha,c}({\rm d}u\times{\rm d}v)=\alpha\int_0^\infty f(u,0)u^{-\alpha-1}{\rm d}u+\alpha c^{-1}\int_0^\infty f(0,v)v^{-\alpha-1}{\rm d}v \quad (33)$$

which is valid provided the integrals are well-defined.

Relation (31) entails

$$\mu^{(t)}(\mathrm{d}x,\mathrm{d}u,\mathrm{d}v) := \sum_{k\geq 1}\varepsilon_{k/t}(\mathrm{d}x)\mathbb{P}\Big\{\frac{|\log W|}{c(t)}\in \mathrm{d}u,\ \frac{|\log(1-W)|}{c(t)}\in \mathrm{d}v\Big\}$$
$$\overset{v}{\to}\ \mathrm{d}x\mu_{\alpha,c}(\mathrm{d}u\times\mathrm{d}v),\quad t\to\infty, \tag{34}$$

where $\varepsilon_{k/t}$ is a probability measure concentrated at k/t, and relation (32) implies

$$\mu^{(t)}(\mathrm{d}x,(0,\infty],\mathrm{d}v)\ \overset{v}{\to}\ \mathrm{d}x\mu_{\alpha,c}((0,\infty]\times\mathrm{d}v),\quad t\to\infty. \tag{35}$$

We use the representation

$$\sum_{k\geq 1}\int_{[0,\infty)\times[0,\infty)}K(z,u,v,k/t)\mathbb{P}\Big\{\frac{|\log W|}{c(t)}\in \mathrm{d}u,\ \frac{|\log(1-W)|}{c(t)}\in \mathrm{d}v\Big\}$$
$$=\int_{[0,\infty)\times[0,\infty)\times[0,\infty)}K(z,u,v,x)\mu^{(t)}(\mathrm{d}x,\mathrm{d}u,\mathrm{d}v)$$

to show that

$$\lim_{t\to\infty}\int_{[0,\infty)\times[0,\infty)\times[0,\infty)}K(z,u,v,x)\mu^{(t)}(\mathrm{d}x,\mathrm{d}u,\mathrm{d}v)$$
$$=\int_{[0,\infty)\times[0,\infty)\times[0,\infty)}K(z,u,v,x)\mathrm{d}x\mu_{\alpha,c}(\mathrm{d}u,\mathrm{d}v), \tag{36}$$

for each $z>0$. The last relation does not follow from (34) automatically, for, with z fixed, the function $K(z,\cdot)$ is not compactly supported on $(0,\infty]\times(0,\infty]\times[0,\infty)$. To prove (36), note that in the same space the function $(u,v,x)\mapsto K(z,u,v,x)1_{\{u\geq 1\text{ or } x>u_n\}}$ does possess a compact support for any fixed $z>0$ (see footnote on p. 12). Hence it is sufficient to check that

$$\lim_{t\to\infty}\int K(z,u,v,x)1_{\{u<1,\,x\leq u_n\}}\mu^{(t)}(\mathrm{d}x,\mathrm{d}u,\mathrm{d}v)$$
$$=\int K(z,u,v,x)1_{\{u<1,\,x\leq u_n\}}\mathrm{d}x\mu_{\alpha,c}(\mathrm{d}u,\mathrm{d}v). \tag{37}$$

To simplify notation we only prove this for $n=1$[2]. With z fixed, the function

$$\hat{K}(z,u,v,x):=\begin{cases}\frac{1-\exp\left(-z(u\gamma_1+\gamma g(x,v))\right)}{u\gamma_1+\gamma g(x,v)}, & u\gamma_1+\gamma g(x,v)\neq 0,\\ z, & u\gamma_1+\gamma g(x,v)=0\end{cases}$$

is continuous and bounded which implies that (37) follows if we prove that

$$u1_{\{u<1,\,x\leq u_1\}}\mu^{(t)}(\mathrm{d}x,\mathrm{d}u,\mathrm{d}v)\ \Rightarrow\ u1_{\{u<1,\,x\leq u_1\}}\mathrm{d}x\mu_{\alpha,c}(\mathrm{d}u,\mathrm{d}v),\quad t\to\infty, \tag{38}$$

[2]The case of general n can be settled by splitting the domain of integration over variable x into segments $[u_{i-1},u_i]$ and checking convergence of integrals over each segment.

and

$$g(x,v)1_{\{u<1,\,x\le u_1\}}\mu^{(t)}(\mathrm{d}x,\mathrm{d}u,\mathrm{d}v)\ \Rightarrow\ g(x,v)1_{\{u<1,\,x\le u_1\}}\mathrm{d}x\mu_{\alpha,\,c}(\mathrm{d}u,\mathrm{d}v),\quad t\to\infty. \tag{39}$$

Vague convergence in (38) and (39) is a consequence of (34). Indeed, if f is continuous and compactly supported, the same is true for $u1_{\{u<1,\,x\le u_1\}}f$ and $g(x,v)1_{\{u<1,\,x\le u_1\}}f$. According to Theorem 30.8 (ii) [1], the vague convergence can be strengthened to the weak convergence if we prove that

$$\lim_{t\to\infty}\int u1_{\{u<1,\,x\le u_1\}}\mu^{(t)}(\mathrm{d}x,\mathrm{d}u,\mathrm{d}v)=\int u1_{\{u<1,\,x\le u_1\}}\mathrm{d}x\mu_{\alpha,\,c}(\mathrm{d}u,\mathrm{d}v) \tag{40}$$

and

$$\lim_{t\to\infty}\int g(x,v)1_{\{u<1,\,x\le u_1\}}\mu^{(t)}(\mathrm{d}x,\mathrm{d}u,\mathrm{d}v)=\int g(x,v)1_{\{u<1,\,x\le u_1\}}\mathrm{d}x\mu_{\alpha,\,c}(\mathrm{d}u,\mathrm{d}v). \tag{41}$$

The prelimiting expression in (40) can be written in the form

$$\sum_{k=1}^{[u_1t]}\int_{[0,1)\times(0,\infty)}u\mathbb{P}\Big\{\frac{|\log W|}{c(t)}\in\mathrm{d}u,\ \frac{|\log(1-W)|}{c(t)}\in\mathrm{d}v\Big\}=\frac{[u_1t]}{c(t)}\mathbb{E}|\log W|1_{\{|\log W|<c(t)\}},$$

which, by (3), converges, as $t\to\infty$, to $\alpha(1-\alpha)^{-1}u_1$, and this is equal to the value of the right-hand side of (40). We now prove (41). Set $\hat{g}(x,v):=g(x,v)1_{\{x\le u_1\}}$ and note that $\hat{g}\in C_K([0,\infty)\times(0,\infty])$. As the function $\hat{g}(x,v)1_{\{u\ge1\}}$ has compact support, equality (41) is justified if we verify that

$$\lim_{t\to\infty}\int\hat{g}(x,v)\mu^{(t)}(\mathrm{d}x,\mathrm{d}u,\mathrm{d}v)=\int\hat{g}(x,v)\mathrm{d}x\mu_{\alpha,\,c}(\mathrm{d}u,\mathrm{d}v). \tag{42}$$

To this end, note that an equivalent form of (42) is

$$\lim_{t\to\infty}\int\hat{g}(x,v)\mu^{(t)}(\mathrm{d}x,(0,\infty],\mathrm{d}v)=\int\hat{g}(x,v)\mathrm{d}x\mu_{\alpha,\,c}((0,\infty],\mathrm{d}v)$$

which holds true by (35). Thus, relation (36) has been proved.

We use integration formula (33) and the equality $g(x,0)=0$ to conclude that

$$\begin{aligned}&\int_{[0,\infty)\times[0,\infty)\times[0,\infty)}K(z,u,v,x)\mathrm{d}x\mu_{\alpha,\,c}(\mathrm{d}u,\mathrm{d}v)\\&\quad=\alpha\int_0^\infty\int_0^\infty\Big(1-\exp\Big(-zu\sum_{i=1}^n\gamma_i1_{\{x\le u_i\}}\Big)\Big)\mathrm{d}xu^{-\alpha-1}\mathrm{d}u\\&\qquad+\alpha c^{-1}\int_0^\infty\int_0^\infty\Big(1-\exp(-z\gamma g(x,v))\Big)\mathrm{d}xv^{-\alpha-1}\mathrm{d}v\\&\qquad\quad=-\log\mathbb{E}\exp\Big(-z\sum_{i=1}^n\gamma_iX_\alpha(u_i)\Big)-\log\mathbb{E}\exp\Big(-z\gamma\sum_m g(t_m,\,j_m)\Big)\end{aligned}$$

for each $z > 0$, which in combination with (36) and (30) gives (28) and thereupon joint convergence (26).

Left with determining the law of $R_{\alpha,\,c}(u)$, fix $\delta > 0$, put

$$R^{(\delta)}_{\alpha,\,c}(u) := \sum_{k\geq 0} 1_{\{S_k \leq u < S_k + \eta_{k+1},\, \eta_{k+1} > \delta\}}$$

and use the equality

$$\mathbb{E} e^{-zR^{(\delta)}_{\alpha,\,c}(u)} = \mathbb{E} \exp\Big(- \int_{[0,\,\infty)} \int_{(\delta,\,\infty]} \big(1 - e^{-z 1_{\{X_\alpha(s) \leq u < X_\alpha(s) + y\}}}\big) \mathrm{d}s \nu_{\alpha,\,c}(\mathrm{d}y)\Big), \quad z \geq 0,$$

which is a particular case of the formula given in [21], p. 136, along with simple manipulations to obtain

$$\mathbb{E} e^{-zR^{(\delta)}_{\alpha,\,c}(u)} = \mathbb{E} \exp\Big(- c^{-1} \int_{[0,\,u]} \big((u-s) \vee \delta\big)^{-\alpha} \mathrm{d}X^{\leftarrow}_\alpha(s)(1 - e^{-z})\Big), \quad z \geq 0.$$

Passing to the limit $\delta \downarrow 0$ and using the continuity theorem for Laplace transforms, we see that

$$\mathbb{E} e^{-zR_{\alpha,\,c}(u)} = \mathbb{E} \exp\Big(- c^{-1} \int_{[0,\,u]} (u-s)^{-\alpha} \mathrm{d}X^{\leftarrow}_\alpha(s)(1 - e^{-z})\Big), \quad z \geq 0.$$

Thus the law of $R_{\alpha,\,c}(u)$ is mixed Poisson with (random) parameter $c^{-1} \int_{[0,\,u]} (u-s)^{-\alpha} \mathrm{d}X^{\leftarrow}_\alpha(s)$. It is shown in [17] that the law of the latter integral is standard exponential. Therefore the law of $R_{\alpha,\,c}(u)$ is geometric with the success probability $c(c+1)^{-1}$, as asserted. The proof of Theorem 1.2 is complete.

6 Proof of Theorem 1.4

By Lemma 2.3, relation (5) follows if we prove that

$$\frac{1 - F(t)}{1 - G(t)} L(e^{ut}) \overset{\text{f.d.}}{\Rightarrow} W_{\alpha,\,\beta}(u), \quad t \to \infty, \tag{43}$$

where $F(t) := \mathbb{P}\{|\log W| \leq t\}$ and $G(t) := \mathbb{P}\{|\log(1 - W)| \leq t\}$, $t \geq 0$.

The first condition in (4) implies $\mathbb{E}|\log W| = \infty$. Hence, in view of Lemma 2.1, it suffices to check that

$$\frac{1 - F(t)}{1 - G(t)} \sum_{k\geq 0} 1_{\{S_k \leq ut < S_k + \eta_{k+1}\}} \overset{\text{f.d.}}{\Rightarrow} W_{\alpha,\,\beta}(u), \quad t \to \infty. \tag{44}$$

By Theorem 2.10 [17] we have

$$\frac{1 - F(t)}{1 - G(t)} \sum_{k\geq 0} \mathbb{E}\big(1_{\{S_k \leq ut < S_k + \eta_{k+1}\}} \big| S_k\big) = \frac{1 - F(t)}{1 - G(t)} \sum_{k\geq 0} \big(1 - G(ut - S_k)\big) 1_{\{S_k \leq ut\}} \overset{\text{f.d.}}{\Rightarrow} W_{\alpha,\,\beta}(u),$$

as $t \to \infty$. The validity of the equality

$$\mathbb{E}\Big(\sum_{k\geq 0}\big(1_{\{S_k\leq t<S_k+\eta_{k+1}\}}-\mathbb{E}\big(1_{\{S_k\leq t<S_k+\eta_{k+1}\}}\big|S_k\big)\big)\Big)^2=\int_{[0,\,t]}G(y)(1-G(y))\mathrm{d}U(y)$$

is easily justified. Further, using the lines of proving Lemma 5.2 [17], one can show that

$$\int_{[0,\,t]}G(y)(1-G(y))\mathrm{d}U(y)\ \sim\ \mathrm{const}\,\frac{1-G(t)}{1-F(t)},\quad t\to\infty.$$

This, Markov's inequality and the conditions of the theorem imply

$$\frac{1-F(t)}{1-G(t)}\sum_{k\geq 0}\big(1_{\{S_k\leq t<S_k+\eta_{k+1}\}}-\mathbb{E}\big(1_{\{S_k\leq t<S_k+\eta_{k+1}\}}\big|S_k\big)\big)\ \overset{P}{\to}\ 0,\quad t\to\infty.$$

Noting that the first multiplier is regularly varying at ∞ and using the Cramér-Wold device we arrive at

$$\frac{1-F(t)}{1-G(t)}\sum_{k\geq 0}\big(1_{\{S_k\leq ut<S_k+\eta_{k+1}\}}-\mathbb{E}\big(1_{\{S_k\leq ut<S_k+\eta_{k+1}\}}\big|S_k\big)\big)\ \overset{\text{f.d.}}{\Rightarrow}\ 0,\quad t\to\infty$$

thereby proving (44) and thereupon (43). The proof of Theorem 1.4 is complete.

7 Proof of Theorem 1.5

Recall the notation $G(t)=\mathbb{P}\{|\log(1-W)|\leq t\}$, $\eta_n=|\log(1-W_n)|$, $n\in\mathbb{N}$, and that $\big(S_n\big)_{n\in\mathbb{N}_0}$ stands for a zero-delayed standard random walk with jumps $|\log W_k|$. Set also

$$q(t):=\sqrt{\mu^{-1}\int_0^t\big(1-G(y)\big)\mathrm{d}y}.$$

Proposition 7.1. *If $\mu=\mathbb{E}|\log W|<\infty$ and condition* (6) *holds, then*

$$\frac{\sum_{k\geq 0}1_{\{S_k\leq ut<S_k+\eta_{k+1}\}}-\sum_{k\geq 0}\mathbb{E}\big(1_{\{S_k\leq ut<S_k+\eta_{k+1}\}}\big|S_k\big)}{q(t)}\ \overset{\text{f.d.}}{\Rightarrow}\ V(u),\quad t\to\infty.$$

Proof. We only prove weak convergence of two-dimensional distributions. The general case is unwieldy and does not require new ideas.

Fix $0<u_1<u_2$. According to the Cramér-Wold device, it suffices to prove that, for any $\gamma_1,\gamma_2\in\mathbb{R}$,

$$\frac{\sum_{j=1}^2\gamma_j\sum_{k\geq 0}1_{\{S_k\leq u_jt\}}\big(1_{\{S_k+\eta_{k+1}>u_jt\}}-(1-G(u_jt-S_k))\big)}{q(t)}\ \overset{d}{\to}\ \gamma_1V(u_1)+\gamma_2V(u_2),\quad (45)$$

as $t\to\infty$. Note that $\gamma_1V(u_1)+\gamma_2V(u_2)$ is a normally distributed random variable with zero mean and variance $\gamma_1^2u_1^{1-\beta}+\gamma_2^2u_2^{1-\beta}+2\gamma_1\gamma_2(u_2^{1-\beta}-(u_2-u_1)^{1-\beta})$. Introduce the σ-algebras $\mathcal{F}_0:=\{\Omega,\oslash\}$, $\mathcal{F}_k:=\sigma\big(W_1,\ldots,W_k\big)$, $k\in\mathbb{N}$ and observe that

$$\mathbb{E}\Big(\sum_{j=1}^2\gamma_j1_{\{S_k\leq u_jt\}}\big(1_{\{S_k+\eta_{k+1}>u_jt\}}-(1-G(u_jt-S_k))\big)\Big|\mathcal{F}_k\Big)=0.$$

Thus, in order to prove (45), one may use a martingale central limit theorem (Corollary 3.1 [13]), according to which it suffices to verify that

$$\sum_{k\geq 0}\mathbb{E}\big(X_{tk}^2|\mathcal{F}_k\big) \overset{P}{\to} \gamma_1^2u_1^{1-\beta}+\gamma_2^2u_2^{1-\beta}+2\gamma_1\gamma_2(u_2^{1-\beta}-(u_2-u_1)^{1-\beta}),\quad t\to\infty, \tag{46}$$

where

$$X_{tk}:=\frac{\sum_{j=1}^2\gamma_j 1_{\{S_k\leq u_jt\}}\big(1_{\{S_k+\eta_{k+1}>u_jt\}}-(1-G(u_jt-S_k))\big)}{q(t)},$$

and

$$\sum_{k\geq 0}\mathbb{E}\big(X_{tk}^2 1_{\{|X_{tk}|>\varepsilon\}}|\mathcal{F}_k\big) \overset{P}{\to} 0,\quad t\to\infty, \tag{47}$$

for all $\varepsilon>0$, hereafter. The inequality $|X_{tk}|\leq \big(|\gamma_1|+|\gamma_2|\big)/q(t)$ reveals that under our conditions relation (47) follows from (46).

It can be checked that

$$\begin{aligned}\sum_{k\geq 0}\mathbb{E}\big(X_{tk}^2|\mathcal{F}_k\big) &= \frac{\sum_{j=1}^2\gamma_j^2\sum_{k\geq 0}1_{\{S_k\leq u_jt\}}(1-G(u_jt-S_k))G(u_jt-S_k)}{q^2(t)}\\ &+ \frac{2\gamma_1\gamma_2\sum_{k\geq 0}1_{\{S_k\leq u_1t\}}(1-G(u_2t-S_k))G(u_1t-S_k)}{q^2(t)}.\end{aligned}$$

Now we prove that

$$\frac{\sum_{k\geq 0}1_{\{S_k\leq u_1t\}}(1-G(u_2t-S_k))}{q^2(t)}=\frac{\int_{[0,\,u_1]}\big(1-G(t(u_2-y))\big)\mathrm{d}\nu(ty)}{q^2(t)} \to u_2^{1-\beta}-(u_2-u_1)^{1-\beta} \tag{48}$$

almost surely, as $t\to\infty$. By the strong law of large numbers for the process $\big(\nu(t)\big)$, the relation

$$\lim_{t\to\infty}\frac{\nu(ty)}{\mu^{-1}t}=y$$

holds almost surely, for all $y\in[0,u_1]$. In addition,

$$\lim_{t\to\infty}\frac{1-G(t(u_2-y))}{1-G(t)}=(u_2-y)^{-\beta}$$

uniformly in $y\in[0,u_1]$. Hence[3]

$$\lim_{t\to\infty}\frac{\int_{[0,\,u_1]}\big(1-G(t(u_2-y))\big)\mathrm{d}\nu(ty)}{\mu^{-1}t(1-G(t))}=\int_0^{u_1}(u_2-y)^{-\beta}\mathrm{d}y=(1-\beta)^{-1}\big(u_2^{1-\beta}-(u_2-u_1)^{1-\beta}\big)$$

almost surely. It remains to note that $\int_0^t\big(1-G(y)\big)\mathrm{d}y\sim(1-\beta)^{-1}t(1-G(t))$.

The next step is to verify that

$$\lim_{t\to\infty}\frac{\int_{[0,\,u_1]}\big(1-G(t(u_2-y))\big)\big(1-G(t(u_1-y))\big)\mathrm{d}\nu(ty)}{q^2(t)}=0 \tag{49}$$

[3]A similar relation in a more general situation can be found in Lemma A.6 [16].

almost surely. To this end, fix $\varepsilon \in (0, u_1)$ and use the monotonicity of $1 - G(t)$ to infer

$$\frac{\int_{[0,\, u_1-\varepsilon]} \big(1 - G(t(u_2 - y))\big)\big(1 - G(t(u_1 - y))\big)\mathrm{d}\nu(ty)}{q^2(t)} \\ \leq \frac{\big(1 - G(\varepsilon t)\big) \int_{[0,\, u_1-\varepsilon]} \big(1 - G(t(u_2 - y))\big)\mathrm{d}\nu(ty)}{q^2(t)}.$$

Relation (48), with u_1 replaced by $u_1 - \varepsilon$, allows us to conclude that the right-hand side of the last inequality tends to zero almost surely, as $t \to \infty$. Further

$$\frac{\int_{(u_1-\varepsilon,\, u_1]} \big(1 - G(t(u_2 - y))\big)\big(1 - G(t(u_1 - y))\big)\mathrm{d}\nu(ty)}{q^2(t)} \leq \frac{\int_{(u_1-\varepsilon,\, u_1]} \big(1 - G(t(u_2 - y))\big)\mathrm{d}\nu(ty)}{q^2(t)},$$

and, as $t \to \infty$, the limit of the right-hand side equals $(u_2 - u_1 + \varepsilon)^{1-\beta} - (u_2 - u_1)^{1-\beta}$. Sending now ε to zero completes the proof of (49). We have thus proved that

$$\lim_{t\to\infty} \frac{\sum_{k\geq 0} 1_{\{S_k \leq u_1 t\}} \big(1 - G(u_2 t - S_k)\big) G(u_1 t - S_k)}{q^2(t)} = u_2^{1-\beta} - (u_2 - u_1)^{1-\beta}$$

almost surely.

Let us show that

$$\frac{\sum_{k\geq 0} 1_{\{S_k \leq ut\}} \big(1 - G(ut - S_k)\big)}{q^2(t)} = \frac{\int_{[0,\, u]} \big(1 - G(t(u - y))\big)\mathrm{d}\nu(ty)}{q^2(t)} \overset{P}{\to} u^{1-\beta}, \tag{50}$$

as $t \to \infty$. Notice first that, for any $\varepsilon \in (0, u)$,

$$\lim_{t\to\infty} \frac{\int_{[0,\, u-\varepsilon]} \big(1 - G(t(u - y))\big)\mathrm{d}\nu(ty)}{q^2(t)} = u^{1-\beta} - \varepsilon^{1-\beta}$$

almost surely, which follows along the lines of the proof of (48). Since the right-hand side of the latter equality tends to $u^{1-\beta}$, as $\varepsilon \downarrow 0$, the proof of (50) will now be completed by showing that

$$\lim_{\varepsilon\downarrow 0} \overline{\lim_{t\to\infty}}\, \mathbb{P}\left\{ \frac{\int_{(u-\varepsilon,\, u]} \big(1 - G(t(u - y))\big)\mathrm{d}\nu(ty)}{q^2(t)} > \delta \right\} = 0,$$

for any $\delta > 0$. By Markov's inequality, the latter relation holds true, if we can check that

$$\lim_{\varepsilon\downarrow 0} \overline{\lim_{t\to\infty}} \frac{\int_{(u-\varepsilon,\, u]} \big(1 - G(t(u - y))\big)\mathrm{d}U(ty)}{q^2(t)} = 0. \tag{51}$$

Using Lemma 8.3 and then the regular variation of $1 - G(t)$ give

$$\int_{((u-\varepsilon)t,\, ut]} \big(1 - G(ut - y)\big)\mathrm{d}U(y) \;\sim\; \mu^{-1} \int_0^{\varepsilon t} \big(1 - G(y)\big)\mathrm{d}y \;\sim\; \varepsilon^{1-\beta} q^2(t), \quad t \to \infty,$$

which proves (51). Therefore, relation (50) holds true.

Finally, an argument similar to that used to establish (49) (or, even simpler, analyzing the asymptotics of expectation) enables us to check that

$$\frac{\sum_{k\geq 0} 1_{\{S_k\leq ut\}}\big(1-G(ut-S_k)\big)^2}{q^2(t)} = \frac{\int_{[0,\,u]}\big(1-G(t(u-y))\big)^2 \mathrm{d}\nu(ty)}{q^2(t)} \overset{P}{\to} 0, \tag{52}$$

as $t\to\infty$. Now convergence in probability stated in (46) is a consequence of (48), (49), (50) and (52) (the last two relations should be used separately for $u=u_1$ and $u=u_2$). $\square$

Now we are ready to prove Theorem 1.5. According to Proposition 7.1, conditions (6) and $\mu=\mathbb{E}|\log W|<\infty$ ensure (it is not necessary to assume here that the distribution of $|\log W|$ belongs to the domain of attraction of a stable law)

$$\frac{\sum_{k\geq 0} 1_{\{S_k\leq ut<S_k+\eta_{k+1}\}} - \sum_{k\geq 0}\mathbb{E}\big(1_{\{S_k\leq ut<S_k+\eta_{k+1}\}}\big|S_k\big)}{q(t)} \overset{\text{f.d.}}{\Rightarrow} V(u), \quad t\to\infty. \tag{53}$$

Assuming further that the assumptions of either of parts (a) through (c) are in force an application of Theorem 2.7 [17] yields

$$\begin{aligned}&\frac{\sum_{k\geq 0}\mathbb{E}\big(1_{\{S_k\leq ut<S_k+\eta_{k+1}\}}\big|S_k\big) - \mu^{-1}\int_0^{ut}\big(1-G(y)\big)\mathrm{d}y}{g(t)}\\ &= \frac{\sum_{k\geq 0}\big(1-G(ut-S_k)\big)1_{\{S_k\leq ut\}} - q^2(ut)}{g(t)} \quad \overset{\text{f.d.}}{\Rightarrow} \quad W_{\alpha,\beta}(u), \quad t\to\infty,\end{aligned} \tag{54}$$

where $\alpha=2$ corresponds to cases (a) and (b), and $g(t)=\sqrt{\sigma^2\mu^{-3}t}\big(1-G(t)\big)$ in case (a) and $g(t)=\mu^{-1-1/\alpha}c(t)\big(1-G(t)\big)$ in cases (b) and (c).
CASES (a), (b1) AND (c1). Our purpose is to demonstrate that

$$\frac{L(e^{ut})-q^2(ut)}{q(t)} \overset{\text{f.d.}}{\Rightarrow} V(u), \quad t\to\infty$$

which is, by Lemma 2.3, sufficient for proving Theorem 1.5 in the cases under consideration.

The assumptions of Theorem 1.5 imply $\mu<\infty$ and $\lim_{t\to\infty} q(t)=\infty$. By Lemma 2.2, the desired convergence follows if we prove that

$$\frac{\sum_{k\geq 0} 1_{\{S_k\leq ut<S_k+\eta_{k+1}\}} - q^2(ut)}{q(t)} \overset{\text{f.d.}}{\Rightarrow} V(u), \quad t\to\infty$$

which, in its turn, is a consequence of (53) and (54) if we still verify

$$\lim_{t\to\infty} g(t)/q(t) = 0. \tag{55}$$

To this end, note first that, in view of Proposition 1.5.8 [3],

$$q^2(t) \ \sim\ \text{const}\, t^{1-\beta}\widehat{\ell}(t), \quad t\to\infty.$$

In case (a) we have $g^2(t) \sim \text{const}\, t^{1-2\beta}(\widehat{\ell}(t))^2$, $t \to \infty$ which implies (55) (note that $\lim_{t\to\infty} \widehat{\ell}(t) = 0$ when $\beta = 0$). In case (b1), $g^2(t) \sim \text{const}\, t^{1-2\beta}(\ell^*(t)\widehat{\ell}(t))^2$, $t \to \infty$, and the validity of (55) is secured by (9). Finally, in case (c1), $g^2(t) \sim \text{const}\, t^{2/\alpha-2\beta}(\ell^*(t)\widehat{\ell}(t))^2$, $t \to \infty$, and (55) is valid in view of (11).
Cases (b2) and (c2). The previous argument allows us to conclude that, first, it suffices to prove that

$$\frac{L(e^{ut}) - q^2(ut)}{g(t)} \overset{\text{f.d.}}{\Rightarrow} W_{\alpha,\beta}(u), \quad t \to \infty,$$

and second, the latter relation is a consequence of the convergence

$$\frac{\sum_{k\geq 0} 1_{\{S_k \leq ut < S_k + \eta_{k+1}\}} - q^2(ut)}{g(t)} \overset{\text{f.d.}}{\Rightarrow} W_{\alpha,\,\beta}(u), \quad t \to \infty. \tag{56}$$

Relation (56) follows from (53) and (54) if we show that

$$\lim_{t\to\infty} g(t)/q(t) = \infty. \tag{57}$$

We only treat case (c2), since the analysis of case (b2) requires similar arguments. By the assumptions of the theorem, $g^2(t) \sim \text{const}\, t^{-2\beta+2/\alpha}\big(\ell^*(t)\widehat{\ell}(t)\big)^2$, $t \to \infty$. Consequently, relation (57) holds automatically if $\beta \in [0,\, 2/\alpha - 1)$ and is secured by equality (12) if $\beta = 2/\alpha - 1$. The proof of Theorem 1.5 is complete.

8 Appendix

Lemma 8.1 is used in the proofs of Lemma 2.1 and Lemma 2.3.

Lemma 8.1. *Let θ be a random variable taking values in $(0,1]$. Then the functions $g_0(y) := \mathbb{E}\exp(-e^y\theta) - \mathbb{E}\exp(-2e^y\theta)$ and $g_4(y) := e^y\mathbb{E}\theta\exp(-e^y\theta)$ are directly Riemann integrable on $\mathbb{R}$, the function $g_1(y) := \mathbb{E}\big(1 - \exp\big(-e^y\theta\big)\big)$ is directly Riemann integrable on the halfline $(-\infty, 0]$, and the functions $g_2(y) := \mathbb{E}\exp\big(-e^y\theta\big)1_{\{\theta > e^{-y}\}}$ and $g_3(y) := \mathbb{E}\big(1 - \exp(-e^y\theta)\big)1_{\{\theta\leq e^{-y}\}}$ are directly Riemann integrable on the halfline $[0,\infty)$.*

Proof. Since the functions g_i, $i = 0, 4$ and g_3 are nonnegative it suffices to check that they are Lebesgue integrable on $\mathbb{R}$ and $[0,\infty)$, respectively, and that the functions $e^{-y}g_i(y)$, $i = 0, 3, 4$ are nonincreasing (see, for instance, the proof of Corollary 2.17 [5]). The first property follows from the equalities

$$\begin{aligned}\int_{\mathbb{R}} g_0(y)\mathrm{d}y &= \int_0^\infty y^{-1}\big(\mathbb{E}e^{-y\theta} - \mathbb{E}e^{-2y\theta}\big)\mathrm{d}y \\ &= \mathbb{E}\int_0^\infty y^{-1}\big(e^{-y\theta} - e^{-2y\theta}\big)\mathrm{d}y = \log 2,\end{aligned}$$

$$\int_{\mathbb{R}} g_4(y)\mathrm{d}y = \mathbb{E}\int_0^\infty \theta e^{-y\theta}\mathrm{d}y = 1$$

and the inequality

$$\begin{aligned}\int_0^\infty g_3(y)\mathrm{d}y &= \mathbb{E}\int_1^\infty y^{-1}\big(1-\exp(-y\theta)\big)1_{\{\theta\le y^{-1}\}}\mathrm{d}y\\ &= \mathbb{E}\int_\theta^1 y^{-1}\big(1-e^{-y}\big)\mathrm{d}y \le \int_0^1 y^{-1}\big(1-e^{-y}\big)\mathrm{d}y<\infty,\end{aligned}$$

where the last chain of estimates is justified by the change of variable and condition $\theta\in[0,1]$ a.s. Further, with $z\in(0,1]$ fixed, the function $y^{-1}(1-e^{-yz})e^{-yz}$ is nonincreasing on $[0,\infty)$. Hence the function $e^{-y}g_0(y)$ is nonincreasing, too. By the same reasoning, with $z\in(0,1]$ fixed, the functions $y^{-1}(1-e^{-yz})$ and $1_{\{z\le y^{-1}\}}$ are nonincreasing on $(0,\infty)$, hence, so are their product and the function $e^{-y}g_3(y)$. The monotonicity of g_4 is obvious.

Since g_1 is nonnegative and nondecreasing, it suffices to show that it is Lebesgue integrable on $(-\infty,0]$:

$$\int_{-\infty}^0 g_1(y)\mathrm{d}y=\int_0^1 y^{-1}\mathbb{E}\big(1-e^{-y\theta}\big)\mathrm{d}y\le\int_0^1\mathbb{E}\theta\mathrm{d}y\in(0,1].$$

The function $h(y):=\exp(-e^y)$ is positive and directly Riemann integrable on $[0,\infty)$. Since g_2 is the convolution of h and the distribution function of $|\log\theta|$, it is directly Riemann integrable on $[0,\infty)$, by Proposition 2.16(d) in [4], p. 297. $\square$

Denote by $\big(S_n^*\big)_{n\in\mathbb{N}_0}$ a zero-delayed random walk with independent increments distributed as a nonnegative random variable ξ^*. Set

$$U^*(t)=\sum_{n\ge0}\mathbb{P}\{S_n^*\le t\},\quad t\in\mathbb{R}.$$

This notation is used in the next two assertions. Although we think Lemma 8.2 may have been known, we give its complete proof as we have been unable to locate it in the literature.

Lemma 8.2. *If $f:\mathbb{R}\to[0,\infty)$ is a directly Riemann integrable function on $[0,\infty)$, then*

$$\overline{\lim_{t\to\infty}}\int_{[0,\,t]}f(t-y)\mathrm{d}U^*(y)<\infty.$$

If f is directly Riemann integrable on $(-\infty,0]$, then

$$\overline{\lim_{t\to\infty}}\int_{[t,\,\infty)}f(t-y)\mathrm{d}U^*(y)<\infty.$$

Proof. If the distribution of ξ^* is non-lattice, an (even stronger) assertion follows from the key renewal theorem. Suppose the distribution of ξ^* is l-lattice, $l>0$. We only treat the case of direct Riemann integrability on $[0,\infty)$.

Since

$$f(t)\le\sum_{n\ge1}\sup_{(n-1)l\le s<nl}f(s)1_{[(n-1)l,\,nl)}(t),\quad t\ge0,$$

we obtain

$$\int_{[0,\,t]} f(t-y)\mathrm{d}U^*(y) \;\le\; \sum_{n\ge 1} \sup_{(n-1)l\le s<nl} f(s)\big(U^*(t-nl)-U^*(t-(n-1)l)\big)$$
$$\le\; U^*(l)\sum_{n\ge 1} \sup_{(n-1)l\le s<nl} f(s)$$

having utilized subadditivity of U^* on $\mathbb{R}$ for the last inequality. It remains to observe that the series on the right-hand side converges, since f is directly Riemann integrable. $\square$

Lemma 8.3 is used in the proof of Proposition 7.1.

Lemma 8.3. *Suppose $f : [0,\infty) \to [0,\infty)$ is a nonincreasing function, $\lim_{t\to\infty} \int_{[0,\,t]} f(y)\mathrm{d}y = \infty$ and $0 < \mathbb{E}\xi^* < \infty$. For $0 \le a < b \le 1$ the following relation holds*

$$\int_{[at,\,bt]} f(t-y)\mathrm{d}U^*(y) \;\sim\; (\mathbb{E}\xi^*)^{-1}\int_{(1-b)t}^{(1-a)t} f(y)\mathrm{d}y, \quad t\to\infty.$$

Proof. If the distribution of ξ^* is nonlattice or 1-lattice, the proof runs the same path as that of Theorem 4 [26] which investigates the case $a = 0$, $b = 1$. If the distribution of ξ^* is l-lattice, the distribution of $l^{-1}\xi^*$ is 1-lattice. Hence, putting $f_l(t) := f(lt)$ we obtain

$$\int_{[at,\,bt]} f(t-y)\mathrm{d}U^*(y) \;=\; \int_{[al^{-1}t,\,bl^{-1}t]} f_l(l^{-1}t-y)\mathrm{d}\sum_{n\ge 0}\mathbb{P}\{l^{-1}S_n^*\le y\}$$
$$\sim\; \frac{l}{\mathbb{E}\xi^*}\int_{(1-b)l^{-1}t}^{(1-a)l^{-1}t} f_l(y)\mathrm{d}y = \frac{1}{\mathbb{E}\xi^*}\int_{(1-b)t}^{(1-a)t} f(y)\mathrm{d}y.$$

$\square$

The statement and proof of Lemma 8.4 which is used to demonstrate Theorem 1.2 retain the notation introduced in Section 5.

Lemma 8.4. *Let $X_t, t>0$ and X be random elements taking values in D, and m_t and m be random point processes taking values in $M_p([0,\infty)\times(0,\infty])$. Weak convergence*

$$\big(X_t, m_t\big) \;\Rightarrow\; \big(X, m\big), \quad t\to\infty \tag{58}$$

under the product topology on $D\times M_p([0,\infty)\times(0,\infty])$ holds if, and only if, for each $f\in C_K([0,\infty)\times(0,\infty])$,

$$\big(X_t, m_t(f)\big) \;\Rightarrow\; \big(X, m(f)\big), \quad t\to\infty \tag{59}$$

under the product topology on $D\times[0,\infty)$.

Proof. Suppose (58) holds. Then, for any fixed function $f\in C_K([0,\infty)\times(0,\infty])$, the mapping $T_f : D\times M_p([0,\infty)\times(0,\infty])\to D\times[0,\infty)$ defined by $T_f(X,m) = \big(X, m(f)\big)$ is continuous in the product topology, and (59) follows by the continuous mapping theorem.

Conversely, suppose (59) holds. Then $X_t \Rightarrow X$ on D, and $m_t(f) \xrightarrow{d} m(f)$, as $t \to \infty$. Consequently, the families $\big(X_t\big)_{t\geq 0}$ and $\big(m_t(f)\big)_{t\geq 0}$ are tight on D and $[0,\infty)$, respectively. Now Prohorov's theorem ensures that these are relatively compact. By Lemma 3.20 [24], the family $\big(m_t\big)$ is tight (hence relatively compact) on $M_p([0,\infty) \times (0,\infty])$. Then the Cartesian product $\big(X_t, m_s\big)_{t,s\geq 0}$ is relatively compact on $D \times M_p([0,\infty) \times (0,\infty])$ which implies that the family $\big(X_t, m_t\big)_{t>0}$ is tight on $D \times M_p([0,\infty) \times (0,\infty])$. It remains to note that all subsequential limits of the collection $\big(X_t, m_t\big)_{t>0}$ are equal in distribution, for (59) holds for any function $f \in C_K([0,\infty) \times (0,\infty])$. $\square$

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